Simple is better. Empirical comparison of American option valuation methods

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1. Introduction

Nowadays, as derivatives become more and more popular, correct and effective pricing gains in importance. Derivation of theoretical values of the instruments has a wide range of applications, from facilitating investing decisions of private investors to enabling correct presentation on the balance sheet of instruments purchased and written by financial institutions. The history of option pricing goes back to the French mathematician Louis Bachelier [1900] who, in his PhD thesis, derived an analytical formula for the price of European call and put options on non-dividend-paying stock. Since then numerous valuation methods have been presented and discussed. In 1973 Fisher Black and Myron Scholes [1973] created a European option pricing formula, which allowed obtaining one option value for all investors, independent of their preferences. Black-Scholes method has many advantages, including simplicity, existence of closed-form solution and availability of data regarding parameters necessary to derive the price. However, it is based on relatively rigorous assumptions, i.e. that the stock price changes follow Wiener process. Implications of these assumptions are used to derive parameters in binomial tree discussed in this article. In parallel to European options valuation methods, since early 1970s American option pricing models have been developed. They may be classified into basic, not necessarily separate, categories: approximation of the American option value with the price of European option, analytical and numerical methods of solving partial differential equations, binomial and trinomial trees, Stochastic Mesh, variety of simulation methods and other, nonstandard methods, e.g. with the use of neural networks. Otherwise valuation methods may be classified depending on the assumption about process driving movements of the price of the underlier. Most often it is Wiener process or various types of jump-diffusion processes. Additionally, some pricing techniques incorporate stochastic character of variables other than underlier's price, as risk-free interest rate or volatility, for example through generalized autoregressive conditional heteroscedasticity (GARCH) models, presented by Bollerslev [1986].

Strategy to exercise a call option on a non-dividend paying stock before its maturity is never optimal. Therefore, for such options any pricing technique appropriate for European options may be used e.g. Black-Scholes formula. For an American call option on a dividend paying stock Black [1975] proposed approximation method in which the higher of two prices obtained from Black-Scholes equation is chosen. Robert Merton [1973] presented alternative derivation of arbitrage pricing differential equation, which allowed to extend the Black-Scholes model for the case of risk-free rate changing over time, dividend payment and especially for options, which may be exercised earlier than maturity. Additionally, he noticed, that arbitrage pricing is possible also when the stochastic process, which describes stock price is almost surely continuous. This enabled the creation of jump-diffusion model in which continuous process is disturbed by large price jumps.

The problem of American options pricing may be reduced to solving partial differential equation with constraints [for the set of constraints for exemplary equation see: Brennan and Schwartz, 1977]. When derivation of the closed-form solution is not possible or complicated, numerical methods are employed. Through approximating partial derivatives of option price by finite differences the equation and constrains may be rewritten in the form allowing for numerical approximation of the solution, using e.g. Euler or Crank-Nicolson method [Crank, Nicolson, 1947]. Various theoretical pricing models may be created assuming different forms of stochastic process describing changes in the price of the underlier, as shown e.g. in Cox and Ross [1976].

In 1979 Cox, Ross and Rubinstein presented an alternative method of incorporating time discretization in the model, through examining changes of the price of the underlier in short discrete periods, creating the binomial pricing method. One of the binomial model's extensions is the trinomial model. Numerous techniques increasing convergence and computational efficiency in tree methods have been presented by, i.a., Broadie and Detemple [1996], Heston and Zhou [2000], Breen [1991] or Figlewski and Gao [1999]. Models allowing to incorporate in the trees changes of interest rates [see: Black, Derman and Toy, 1990] and volatility [e.g. in: Ho, Stapleton and Subrahmanyam, 1995] over time, including GARCH effect for variance [Ritchken, Trevor, 1999] have also been introduced. Another method of American options pricing, based on building the mesh of underlier's prices has been presented [1997] and extended [2000] by Broadie and Glasserman.

One of the first authors who employed simulation in option pricing was Boyle [1977] who valued European options using Monte Carlo method. The first using simulation for pricing options, which might be exercised before maturity, applying backward induction, was Tilley [1993]. Carriere showed that the choice of the decision rule regarding early exercise or keeping the option can be modeled through estimation of the series of conditional expected values [Carriere, 1996]. Tsitsiklis and Van Roy [2001] and Longstaff and Schwartz [2001] presented algorithms based on regression that allow to estimate these values. In the model created by Longstaff and Schwartz paths of the price of the underlier are generated by simulation. Then, the expected value of continuation at each moment is estimated by the least squares method and so the algorithm is called Least Squares Monte Carlo (LSM). The advantage of simulation techniques is the fact that they may be used to value options when its price depends on the value of more than one asset. They are also successfully employed to derive values of the options, for which payment depends on stock price trajectory and these, which may be exercised before maturity. Furthermore, the Monte Carlo method allows to assume practically any process for the price of the underlier [see in: Cox, Ross, 1976], e.g. jump-diffusion process [Merton, 1976].

Constant volatility option pricing models have been extended to incorporate heteroscedasticity of variance in time. The most popular of these methods is the Stochastic Volatility (SV) model by Heston [1993] for continuous time and, allowing for modeling in discrete time, various GARCH-based models. One of the first to present theoretical models of stochastic volatility have been Hull and White [1987] and Wiggins [1987]. Later studies include Bakshi, Cao and Chena [1997] and Bates [2000]. In any SV model estimation is complicated by the fact that the volatility is unobserved. One of the methods to estimate volatility is Markov Chain Monte Carlo [for detailed description refer to: Jacquier, Polson and Rossi, 2004]. In case of European options the common advantage of SV models over GARCH is the existence of closed-form formula for the option price, in which the volatility has to be substituted. This advantage does not apply to American options. Moreover, in many cases GARCH models give similar results to SV models, as shown in Nelson [1990] or Duan [1997].

This article compares two option pricing techniques: commonly used but based on restrictive assumptions simple binomial model and GARCH-LSM, less computationally effective but allowing to incorporate the fact that the volatility of the underlier is not constant in time. The second of the mentioned models is by itself an interesting example of adaptation of GARCH dynamics to valuation through combination of simulation and least squares method. The methodology applied is based on the idea of Stentoft [2004] to combine the method presented by Duan [1995] for European options with Least Squares Monte Carlo proposed by Longstaff and Schwartz [2001] in order to incorporate the possibility of early exercise.

Additionally to the comparison of estimations obtained from both methods with market prices, the possible impact of option characteristics on the level of pricing errors was studied. The characteristics considered were time to maturity reflected in the number of simulation steps and binomial tree levels and “moneyness” ratio. It is to be supposed that the more the option is *out-of-the-money* the higher the risk associated with purchasing it. The intuitive hypothesis that valuation errors might then be higher is verified. Because of the time covered in the study, it is possible to make an attempt to verify if the increased volatility associated with financial crisis affects the purposefulness of using one of the methods in comparison with the other. Furthermore, the result stemming from previous studies indicate that incorporating discrete dividend in the binomial model shall not significantly influence the pricing results is verified. The article is organized as follows. Section 2 discusses option pricing methodologies for binomial tree and GARCH-LSM model. In Section 3 detailed results of empirical study are presented and discussed. The last section summarizes the main findings and concludes the paper.

2. Option pricing methodology

2.1. Binomial tree

Binomial tree is the simplest and most commonly used in practical applications method of American options pricing. John Cox, Stephen Ross and Mark Rubinstein invented the model. The assumptions underlying the method, except time discretization and incorporating possibility of exercise before maturity are the same as for the Black-Scholes model for European options. Namely, it is assumed that there is no arbitrage possibility, no transaction costs nor taxes; interest rate is constant and short selling is available. Changes of the price of the underlier over time are discrete, in every period there are two possible scenarios: upward movement (price increases u times) or downward movement (price drops $\frac{1}{d}$ times), the probability of each scenario is constant and the number of periods is finished. Assumption of discrete amount of moments in time when American option may be exercised equals to approximating American option with its Bermudan counterpart.

A binomial tree is a practical and generally accepted technique of option pricing. For simplicity it is assumed for now that the underlier does not pay dividend. There are a few methods that allow to incorporate dividend in the model, presented further. The tree is a diagram representing possible different paths of the price of the underlier in a time span from a given pricing date to expiry, as presented in Figure 1. The time from the moment of valuation till option expiry is divided into a number of short, equal periods of time. The length of each period is denoted by Δt. The point on the diagram furthest to the left depicts current option price S_0. For a given moment t the price equals S_t and in every subsequent moment it changes either to dS_t (with probability $1 - p$) or uS_t (with probability p), where $0 < d < 1 < u$, and t is any moment before option expiry. On the i-th level of the tree, corresponding to moment $i\Delta t$, there are $i + 1$ nodes, each of which contains possible price of the underlier. Stock price in the j-th node at the i-th level of the tree equals $S_0u^jd^{i - j}$, for $j = 0, \ldots, i$. In order for the subsequent increase and decrease to compensate, it is assumed that $d = \frac{1}{u}$. Moreover, for the risk free interest rate r holds: $u > 1 + r > d$, otherwise the no arbitrage condition would not be met.

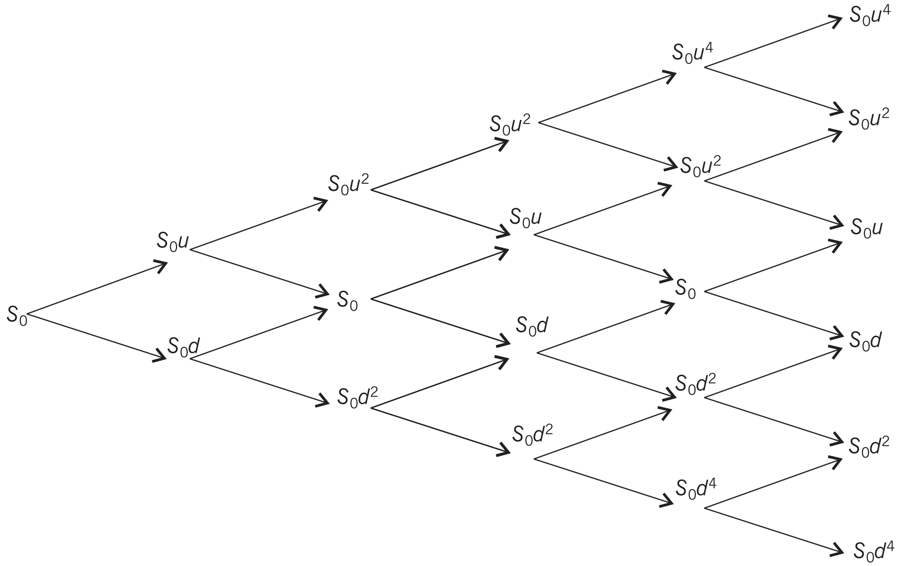

Figure 1.
Binomial tree
Source: Adopted from Hull [1993, p. 338].

Parameters of both binomial method are calibrated as if change of the rate of return of the underlier was a discrete approximation of the Black-Scholes method, i.e. values of the parameters u and d are obtained by equating the mean and variance of the stock price at time $t + \Delta t$ in binomial model to the parameters of the geometric Brownian motion, under assumption that current moment is t. The value of the price of the underlier at time $t + \Delta t$ – random variable $S_t + \Delta t$—in the binomial tree is equal to S_tu with probability p and S_td with probability $1 - p$. The expected value of $S_{t + At}$ is consequently equal to:

$$E\left(S_{t+1}\right) = pS_tu + (1 - p)S_td \qquad [1]$$

and the variance is:

$$Var\left(S_{t+1}\right)=E\left(S_{t+1}^2\right)-E\left(S_{t+1}\right)^2=pS_t^2u^2+(1-p)S_t^2d^2-\left[S_tpu+S_t(1-p)d^2\right] \quad [2]$$

The expected value and variance, under the assumption that the price of stock changes according to geometric Brownian motion are equal, respectively: $S_te^{r\Delta t}$ and $S_t^2\sigma^2\Delta t$, where σ is the volatility. Assuming continuous compounding and $u=\frac{1}{d}$ by equating moments following formulas for u, d and p are obtained [see: Hull, 2004]:

$$u=e^{\sigma\sqrt{\Delta t}} \quad [3]$$

$$d=\frac{1}{u} \quad [4]$$

$$p=\frac{e^{r\sqrt{\Delta t}}}{u-d} \quad [5]$$

From the equations above it is evident that the price of the underlier can decline or increase from one moment to the next proportionally to the length of time period between subsequent moments and the volatility. Formulas [4], [5] and [6] are sufficient to unambiguously define a binomial tree.

When pricing an option with the built model of binomial tree, the value of, say put, option, can be obtained with the use of backward induction, starting at the moment of expiry. In a moment when the option expires the price is known—it is equal to the exercise value, for a put option max(K – S_T, 0), where K denotes strike price and T moment of maturity. In a single induction step the option is priced as follows:

Let us assume that the option price for the nodes after moment t has already been determined. For a moment t – Δt first the expected value of the price at moment t, discounted at a risk free interest rate, has to be calculated. For a given node let V_d and V_u be option values at the ends of two later nodes. The option will be worth V_u with probability p and V_d with probability 1 – p. Discounted expected value of the option equals $e^{-r\Delta t}\left(pV_u+(1-p)V_d\right)$. This would be the value of the option's price at the moment t – Δt if not for the possibility of early exercise at this moment. In case of an American option it is necessary to verify if immediate exercise will not prove more profitable than retaining an option to the next period. If the cash flow from the exercise at t – Δt: max(0, K – $S_{t-\Delta t}$ is higher than expected payment from keeping the option, then this cash flow is taken as an option value in the given node. Therefore, the option price at time t – Δt equals to:

$$\max e^{-r\Delta t}\left[pV_u+(1-p)V_d\right];\max\left[0;K-S_{t-\Delta t}\right] \quad [6]$$

For the call option the procedure is analogous in principle. The cash flow from the exercise at $t - \Delta t$ differs, naturally, and is equal to $\max(S_T - K, 0)$. For call option on non-dividend paying asset early exercise is never optimal so there is no need to consider the profitability of immediate exercise.

Moving successively through all the nodes at all of the tree levels it is possible to obtain the current option value. This value includes both different possible price paths and the possibility of early exercise of an American option. The description of the pricing procedure is based on the assumption that all investors are risk neutral. However, the method is correct also without making any assumptions about investors' preferences. It may be shown through alternative interpretation and derivation with the use of replicating portfolio.

For options on dividend paying stocks, the option exercise price is decreased by the dividend value only when unusual one-time dividends are paid. It is not altered at the dividend date in case of normal, quarterly dividend. This stems from the fact that investors will have assimilated the information about the amount of the dividend before it is paid in their decisions regarding submitted prices to buy and sell the option. As a result the market option premium will adjust accordingly. Therefore, to price an option on a dividend paying stock it is necessary to include impact of the dividend on the underlier price from the moment when its amount is known.

Theoretically, the value of stocks of a given company comprises the values of all the assets which this entity owns. At the moment when the company pays dividend, its assets value decreases by the dividend amount multiplied by the number of stocks. Consequently, the price of each stock shall be then decreased by the dividend value associated with a single stock. This holds under the assumption that the amount of money at company's balance sheet is priced exactly the same as the same amount on the stockholder account. In practice the decrease in opening price on the day after dividend payment is slightly lower than the dividend amount due to taxes. Further, for simplicity, the term "dividend" describes the amount of stock price decrease as a result of the establishment of dividend payment. Ex-dividend date is the first market day when action buyer no longer receives the right to the nearest dividend; it will be paid to the current stockholder. This usually take place somewhat earlier than the date when the dividend is actually paid (i.e. credited to brokerage accounts of stockholders of the company). It may be assumed that the dividend amount will be discounted by the investors in stock price before the ex-dividend date. Therefore, further on, i.e. in the empirical application of binomial model, the notion "dividend payment date" refers to ex-dividend date. In case the anticipated dividend value is higher than the stock price generated in the model on a given moment, it is assumed that asset value decreases to 0, so that the stock price remains nonnegative [following the approach in: Nieuwenhuis, Vellekoop, 2006].

As a consequence of the mentioned stock price change at the dividend payment date, the assumption that the result of subsequent increase and decrease in the underlier price is the same as in the case of price decrease and then increase. If the dividend D is paid, say, at the moment S_{t+1}, then $(S_t u - D)d \neq (S_t d - D)*u$. This means that the binomial tree built for dividend paying stock will not recombine, as illustrated in Figure 2. The property of recombination significantly decreases the number of nodes in the binomial tree and thus simplifies the calculations. Incorporating the dividend directly in pricing by decrease of stock price in each node at the dividend payment date by the dividend amount may cause considerable growth of the binomial tree and in consequence complications in determination of the option price. There are many ways to modify the binomial method to incorporate dividend and preserve the tree recombination characteristic at the same time.

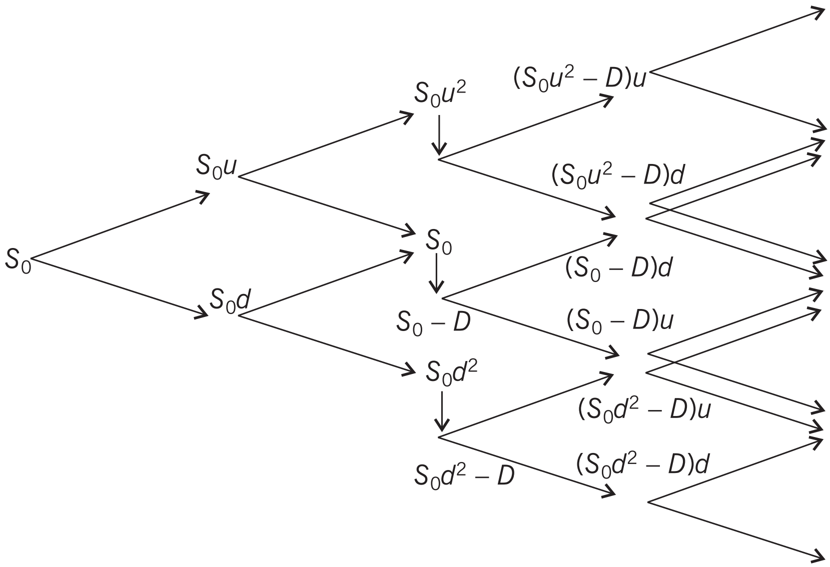

Figure 2.
Standard binomial tree does not recombine after the dividend payment date
Source: Adopted from Hull [1993, p. 347].

For options on indexes, currency and bucket options continuous dividend rate q paid over time is taken [see: Hull, 1993]. Stocks usually pay dividends at intervals. It may be approximately assumed that the value of dividend is a certain percentage of the stock price at the dividend payment date. The binomial tree modified this way, as illustrated on Figure 3, has the property of recombination and pricing remains arbitrage pricing [see: Cox, Ross and Rubinstein, 1979]. The basis is the assumption that substantial changes in stock value could affect the decision of a company regarding dividend payment or its amount. In practice, the dividend value is usually known in advance. In case of unfavorable stock price movements or other adverse events the firm might want to change the dividend amount or abandon payment. However, this happens very rarely as companies want to remain reliable in stockholders' eyes, so simple proportional dividend method is rather unrealisic.

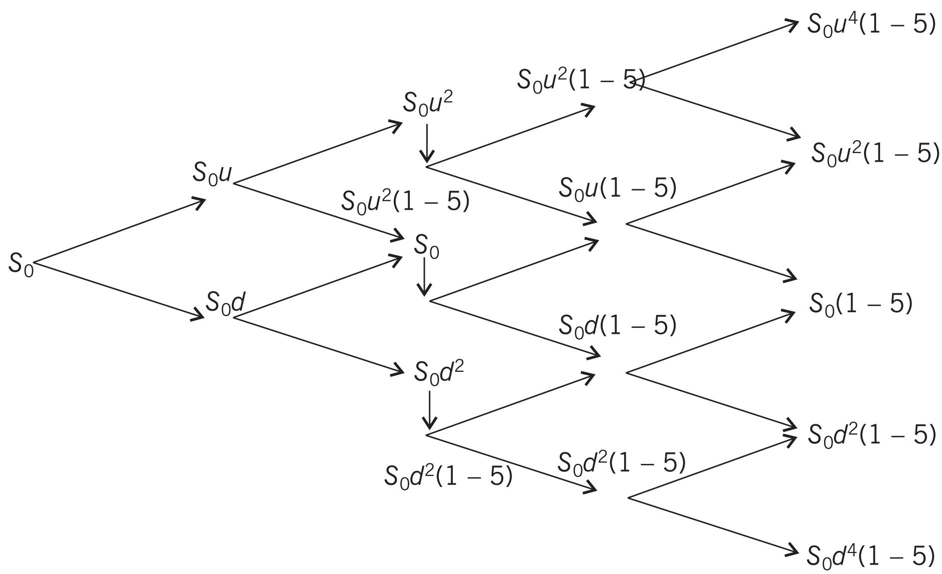

Figure 3.

Binomial tree when stock pays proportional dividend

Source: Adopted from Hull [1993, p. 349].

Another method of incorporating dividend payment in the model is closer to reality, as it assumes discrete dividend value denominated in monetary units. Dividend amount is also supposed to be known. The method is based on dividing the stock price into two separate components. The first one is the variable "risky" part. The second component is the value of future dividends discounted with the risk free rate. When building a binomial tree, the value in the initial node is equal to the current price of the asset decreased by the present value of the dividends paid by the stock until maturity. Further the tree is built in the standard way [description of the method may be found in: Hull, 1993]. An alternative method is to calculate future value of the dividends at option maturity using risk free interest rate and adding it to the strike price [see: Musiela, Rutkowski, 1997]. A combination of both approaches was presented by Bos and Vandermark [2002]. They suggest to calculate the future value of the dividends paid on dates closer to maturity and to incorporate them by adjusting the strike price and discount the dividends paid during period closer to present and subtracting their present value from the stock price at the initial node of the binomial tree. The main weakness of all three models is the fact that for options with different maturities a different number of future dividends will be incorporated when building a binomial tree. As a result binomial trees built for options with different maturities but other characteristics the same will depict different price processes of the same underlier. In order to prevent unrealistic dependence of the changes of stock price from options which are written on this stock, all known future dividend values could be incorporated in the binomial tree. However, then [as noted in: Nieuwenhuis, Vellekoop, 2006] values of the dividends paid after option's maturity would have influence on its value, which is not the case in reality, either. Furthermore, bringing dividend amounts to one or two moments in time results in incorporating in the model their value but not their influence on profitability and reasonableness of early exercise. Despite these inaccuracies the method of bringing dividends to two moments in time is often used in practice. Main reason for that is the fact that it yields similar results as more complicated methods, e.g. Nieuwenhuis-Vellekoop method [2006], and is relatively simple to implement.

2.2. GARCH-LSM model

One of the first and most popular option pricing models is the Black-Scholes model for European options. It is based, i.a. on the assumption of continuous time and particular price process for the underlier. Under these restrictive assumptions the market is complete—it is possible to make a replicating portfolio for any derivative. This allows for the use of property of risk neutrality in pricing under no arbitrage opportunities condition. Results obtained by Black and Scholes can be generalized. Let P be the real probability measure. Harrison and Kreps showed that on the market there are no arbitrage opportunities if there exists martingale measure Q equivalent to P. Then the present value of derivative can be obtained by discounting its expected value (calculated with respect to measure Q, i.e. $E^{Q}(x)$) with the risk-free interest rate. Furthermore, the market is complete if there exists exactly one such measure. As there exists only one such measure, the price of the instrument is designated unequivocally for all investors, i.a. it does not depend on investors risk preferences.

When conditional variance which is not constant in time is introduced, as it is the case in GARCH models, the perfect replication argument no longer holds. The market is not complete and there may exist many martingale measures equivalent to P. The traditional pricing method—with the risk neutrality approach, used e.g. in binomial trees, cannot be used. Jin-Chuan Duan [1995, p. 14] mentions studies containing examples when stock price changes are described by the GARCH process and in consequence the value of the derivative differs depending on investor and it is not possible to employ martingale pricing. Therefore, he presents a modification of the risk neutrality property which gives a theoretical basis for using risk neutral interest rates when pricing options and at the same time incorporating GARCH process for the variance. This extension of the risk neutral valuation was called "Locally Risk-Neutral Valuation Relationship"—LRNVR.

In the model presented by Duan the equation for the current expected value of the rate of return is not purely autoregressive but it takes the following form:

$$r_t = r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2\varepsilon_t \qquad [7]$$

Where the rate of return r_t is the dependent variable, and r is the risk-free rate. Innovations ε_t are described by the GARCH-M process:

$$\varepsilon_t | F_{t-1} \; N\left(0, \sigma_t^2\right) \qquad [8]$$

$$\sigma_t^2 = \omega\sum_{i=1}^{q}\gamma_i\varepsilon_{t-1}^2 + \sum_{i=1}^{p}\beta_i\sigma_{t-1}^2 \qquad [9]$$

It is assumed that parameters γ_i and β_i are nonnegative and $\omega > 0$, to exclude events when estimated variance would be negative and the assumption $\gamma_i + ¿\sum_{i=1}^{p}\beta_i < 1$ $\sum_{i=1}^{q} ¿$ guarantees stationarity of the process. In the special case when $p = 0$ and $q = 0$ the conditional variance is constant over time and for European options method becomes analogous to the Black-Scholes model and in general, as for the theoretical assumptions, also to the binomial model.

The rate of return in the mean equation is dependent on variance so presented model is the GARCH-M. With some approximation the GARCH-M model can be treated as ARMA model for variance [Bollerslev, 1986] and so it may help in explaining volatility clustering—a phenomenon common for financial assets when periods of high and low variance of rate of return are observed in turns. Further the notion of GARCH will refer to GARCH-M model in the form presented above. In empirical study estimation GARCHM(1, 1) model was used. The l parameter may be interpreted as risk premium. The higher its value, the more investor gains or loses in case of respectively increase or decrease of risk expressed as volatility in the underlier—σ_t.

In the equation [7] asset's rate of return r_t depends on risk preferences. For that reason using the model in the form in which it is currently presented, without any further assumptions, would result in obtaining derivative value also dependent on preferences. In order to derive unequivocal price it is necessary to transform the asset price process, by changing the measure, so that for every moment in time, expected value of the price of the asset be equal to the risk-free rate. To make it possible, assumptions about risk aversion of the investors and form of their utility function have to be made or, alternatively, about the linearity of the utility function [for detailed description of the conditions see: Duan, 1995]. Duan [1995] proves that if these conditions are met, there exists martingale measure Q satisfying the definition of LRNVR:

1. It is equivalent to the real measure P,
2. $\frac{X_t}{X_{t-1}} \vee F_{t-1}$ has lognormal distribution under Q,
3. $E^Q\left(\frac{X_t}{X_{t-1}} \middle| F_{t-1}\right) = e^r$ and
4. $Var^Q\left[\ln\left(\frac{X_t}{X_{t-1}}\right)\middle| F_{t-1}\right] = Var^P\left[\ln\left(\frac{X_t}{X_{t-1}}\right) \vee F_{t-1}\right]$ almost surely with respect to measure P.

If the LRNVR is satisfied the local martingale pricing is possible. Conditional expected value of the asset's rate of return with respect to measure Q, directly from the 2nd condition of the definition is equal to risk-free rate. Using the relationship in 4. and available data it is also possible to estimate conditional variance with respect to measure Q. From the definition of local risk neutrality this variance is equal to the conditional variance with respect to measure P, which can be directly estimated. The transformed process of the underlier with respect to measure Q is described by the system of equations:

$$r_t = r - \frac{1}{2}\sigma_t^2 + \xi_t \qquad [10]$$

$$\xi_t \middle| F_{t-1} N\left(0, \sigma_{t-1}^2\right) \qquad [11]$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \gamma_i \left(\xi_t - \lambda\sigma_t\right)^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-1}^2 \qquad [12]$$

Pricing under the LRNVR is not equivalent to the standard martingale pricing. One of the main differences is presence of the λ coefficient in the conditional variance equation. It means that the risk neutralization is merely local, whereas global risk premium influences the conditional variance. Consequently, Locally Risk-Neutral Valuation Relationship may be satisfied even if, as it happens in case of GARCH process, unconditional variance of the underlier and its conditional variance for more than one period is not constant when the measure is transformed to equivalent martingale measure. It is sufficient that conditional variance for one moment ahead remains constant when the measure is changed.

The proof of LRNVR presented in Duan [1995] requires the assumption that conditional distribution of innovations is normal. Generally this assumption is not necessary to develop the pricing method when the rate of return dynamics is described by the GARCH process. Generalizations allowing the use of different parametrical distributions of residuals can be found in papers by Siu, Tong and Yang [2004] or Hafner and Herwartz [2001].

Initially LRNVR was used for European options pricing but it may serve as a theoretical basis also for valuation of American options. In such case it is necessary to consider the possibility of early exercise if it proves to be optimal. Lars Stentoft [2004] suggested to do it by using a modification of the Least Square Monte Carlo (LSM) algorithm. Following the approach presented in his article, here the LSM model specification proposed by Longstaff and Schwartz [2001] is used. Similar techniques may also be found in earlier papers e.g. Carriere [1996] and Tsitsiklis and Van Roy [2001], but they do not present as many practical examples of method implementation.

LSM is one of the Monte Carlo valuation methods. The price of derivative is obtained by averaging value estimated for many generated paths of price of the underlier. First, with the aid of Monte Carlo method, N trajectories of the stock price are generated. Each path contains of $T + 1$ stock price from the current moment until maturity, at daily intervals. When the LSM method is adapted to valuation, under assumption that volatility of the underlier is described by GARCH process, as in empirical part of this paper, price trajectories are simulated using equations [10], [11] and [12], then $S_{t+1} = S_t * e^{r-\frac{\sigma}{2}+\xi_t}$.

In the case of American options, the early exercise opportunity exists and usually the optimal moment to exercise American option is not known. The strategy regarding possible continuation is determined by using the least squares method. In order to determine if on a given moment t exercising is an optimal strategy for the investor, one has to compare payment in case of immediate exercise with conditional expected value of continuation, conditional on currently available information. Investor will exercise an option before maturity if expected payment from continuation is lower than the payment in case of immediate exercise. Therefore, value of an option at the moment t is equal to:

$$V_t = \max\left[e^r E\left(V_{t+1} \vee F_t\right), H_t\right] \qquad [13]$$

where H_t is the payment from immediate exercise:

$$\begin{gathered} \max\left(0; K - S_t\right) \text{ for call option} \\ \max\left(0; S_t - K\right) \text{ for call option} \\ H = ¿ \end{gathered} \qquad [14]$$

F_t denotes all the information available at time t.

For the binomial tree, as it was shown earlier, the option price may be obtained basing directly on the above equation, as the value $E\left(V_{t+1} \vee F_t\right)$ is known as a result of discounting with the use of transition probabilities. In the case of Monte Carlo simulation, the situation is more complicated as the mentioned expected value is not known—it has to be approximated. Longstaff and Schwartz [2001] presented a simple algorithm to approximate the expected value, based on the least square method.

At maturity, payment from exercising the option is known. Working backwards from that moment, at each step the expected value of continuation (i.e. not exercising the option at the given moment) is estimated using the least squares method from all data regarding future price of the underlier available from the simulation. The regression of realized ex-post payments from continuation of functions of the values of state variables is conducted. In practice the only state variable used is price of the underlier at a given moment. The value of continuation can be presented as linear combination of functions of the stock price at time t:

$$E\left(V_{t+1} \middle| F_t\right) = \sum_{k=1}^{K} a_k \varphi_k\left(S_k\right) \quad [15]$$

$\varphi_k(S_k)$ for i from 1 to K is a set of basis functions (deterministic). Implicit is the assumption that the function space, from which stems the function describing the expected value of continuation. is known. The parameters a_k are estimated by the least squares method.

As the basis functions any weighted polynomials may be used, e.g. Legendre, Laguerre, Czebyszew or Hermite polynomials may be used. Clément, Lamberton, and Protter [2002] proved that estimated conditional expected value of continuation converges with probability one to the real expected value when the number of the basis functions approaches infinity. However, according to the results presented by Longstaff and Schwartz [2001], adding more polynomials does not improve significantly numerical results. Using four first basis functions is sufficient to obtain effective convergence of the algorithm for American options. Therefore, in empirical analysis, for simplicity, constant and polynomials X, X^2, X^3 were used. Additionally, in regression only the price paths for which option was in-the-money were exploited because, as mentioned by the authors of the article, it increases algorithm efficiency and decreases time complexity of calculations.

Estimated values are approximations of the value of the conditional expected value function. This way, for each possible moment when option could be exercised the full specification of optimal exercise strategy along every possible path of underlier's price is obtained. Consequently, using backward induction, the current price of the option is derived.

3. Empirical comparison of pricing methods

An empirical comparison of binomial and GARCH-LSM methods has been conducted for American vanilla put and call options on stocks of ten enterprises, one of which has been analyzed in more detail, including dependence of pricing errors on option characteristics—time remaining to maturity and "moneyness" ratio. Furthermore, an example of one enterprise was chosen to assess the impact of incorporating dividends in binomial tree on pricing results. This chapter contains results of the analysis and discussion thereof.

3.1. Data

For estimation of the GARCH-LSM model and comparison with the binomial tree current data regarding options from a five-year period from 01.11.2006 to 01.11.2011 was used. Records, for which market option price was not given, was equal to 0 (the case which was regarded as lack of transaction conclusion, also verified by checking the transaction volume) or these for which time to maturity was shorter than 1 day (less than spot) were removed from the initial data set. Afterwards, according to the approach proposed by Stentoft [2005], contracts that did not satisfy conditions arising from no arbitrage assumptions presented by Hull [1993], i.e. these for which option price was higher than current stock price or equation: $V_C - V_P < S_t - Ke^{-r(T-t)}$, where V_C, V_P denote values of call and put option respectively, was not satisfied, were also deleted. These cases were regarded as proof of the market not liquid enough to take advantage of arbitrage opportunities. The price on such market may not be a reliable reflection of the true instrument value.

The initial number of observations as well as the number used in the final study is presented in Table 1. in the Appendix. For analysis of dependence of pricing accuracy on options characteristics the set of observations for one company had to be subdivided. Therefore, data set with the highest number of records—options on Apple stocks—was chosen. Another reason for choosing this data set was the comparison of statistics regarding option trading in the studied period. Apple options trading volume in 2006–2008 was the highest in comparison to other stock options according to CBOE statistics[1]. In subsequent years these options were also one of most often bought and sold options. Higher trading volume, and so higher liquidity of an instrument results in more reliable market price (in terms of being driven by market supply and demand forces). The more buy and sell transactions are concluded, the closer the transaction price is to the fair price. Therefore, other nine companies for which stocks options were studied were also chosen from the most often traded in the given period.

The characteristics studied were "moneyness" ratio and exceptionally long and short time to maturity. The data set of options on Apple stocks was distributed into four classes depending on time remaining to maturity—0 to 2 months, 2 to 6 months, 6 to 12 months and above one year. Another division depends on "moneyness" ratio; here five ranges were created: above 80% (deep-in-the-money), 5–80% (in-the-money—ITM), from –5% to 5% (at-the--money—ATM), from –80% to –5% (out-of-the-money—OTM) and below –80% (deep-out-of-the-money). Number of put and call options in each class are presented in Table 3. in the Appendix.

"Moneyness" ratio expresses distance of the strike price from current price of the underlier, so for call option it is calculated as the difference be-

[1] CBOE Holdings Inc., *CBOE market statistics*, http://www.cboe.com/data/AnnualMarketStatistics.aspx.

tween current value of the underlying asset and strike price divided by current price of the underlier (contrary to commonly used definition as quotient of spot price and discounted strike price, see for example [Kokoszczyński, Sakowski and Ślepaczuk, 2010]). For put option the ratio is equal in absolute value and the opposite sign as for the call option with identical characteristics.

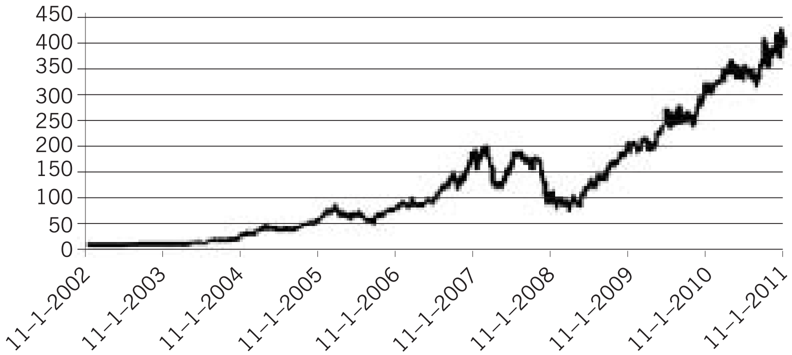

Figure 4.

Apple stock prices from 01.11.2002 to 01.11.2011

Source: Based on data from Yahoo!Finance service.

As the Apple stocks in the given period did not pay dividend, in order to measure influence of incorporating it in the model, options on Bank of America stocks were used. Table 4. in the Appendix shows dividends paid by BAC in the studied period. Data regarding dividend value, ex-dividend date and date when rights to the dividend were announced were obtained from Bank of America website[2].

Data regarding stocks was obtained from Yahoo!Finance service. Prices are dividend- and split-adjusted, i.e. in case when stocks are divided or other event takes place that changes the stock value momentarily, as dividend, merge, restructuring or liquidation, the stock price is adjusted accordingly.

The series stock prices is available from up to 1984. It was assumed, that information contained in the data from such distant past are not highly related to current changes on the market and incorporating whole historical time series could decrease instead of increasing estimates precision. Thus, for GARCH estimation quotes from 01.11.2002 to 01.11.2011 were used. The data range was chosen this way on purpose in order not to omit important information nor increase estimation errors and at the same time to minimize computational time. For each date for which option premiums are available,

[2] Bank of America, http://investor.bankofamerica.com/phoenix.zhtml?c = 71595&p = irol--dividends_pf.

the GARCH model is estimated based on four year series of stock prices preceding given date.

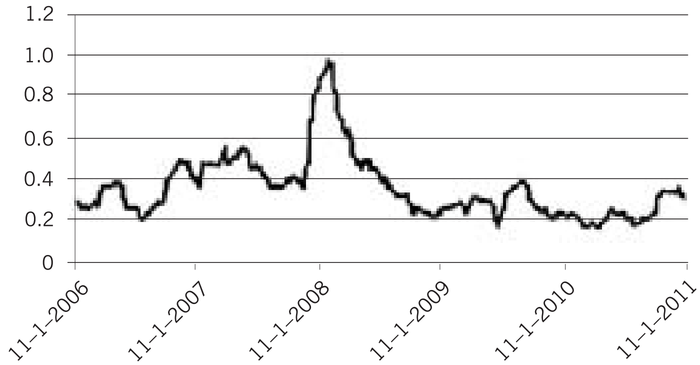

Figure 5.

Annualized historical volatility of Apple rates of return from 01.11.2008 to 01.11.2011
Source: Based on data from Yahoo!Finance service.

From time series of stock prices logarithmic rates of return were calculated. For a moment t logarithmic rate of return is equal to:

$$r_t = \ln\left(\frac{S_t}{S_{t-1}}\right) \qquad [16]$$

For the purpose of option pricing it is common to take rate paid by government bonds or LIBOR as a proxy of risk-free interest rate. For valuation of single instrument interest rate time structure may be considered. However, in most empirical studies regarding the usage of pricing models, if they are not designed to test the impact of interest rates term structure on obtained estimates, constant risk-free interest rate over the life of the derivative is assumed. Here, due to large data set and a relatively high computational time of the studied method, for simplicity constant risk free rate was used. Data regarding one month LIBOR was obtained from Reuters.

As an approximation of volatility in the binomial tree, historical volatility of stock prices on a given date calculated based on 50 last rate of return values was taken:

$$\hat{s} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(r_i - \acute{r}\right)^2} \qquad [17]$$

with $n = 50$.

The data range includes the period of financial crisis and related fluctuations of market prices and so also occurrence of higher values of and changes in stock price variance. It allows to verify the hypothesis that pricing incorporating the GARCH model for the underlier yields better results than other methods when volatility is clearly not constant in time. High Apple rate of return volatility from end of September to November 2008 accompanied, as it is often observed for financial time series, drop in Apple stock price (see Figures 4. and 5.). This decrease was partly linked to general mood on the market related to the crisis, firstly i.a. as a consequence of rejection of Emergency Economic Stabilization Act by the United States House of Representatives. A crisis causes also decrease of demand for more expensive technology goods which constitute significant share of Apple products. Analysts, apprehending that this may impact company's revenue, lowered its ratings. This in turn influenced investors desire to own Apple stocks, resulting in drop in their value. When reality proved that analysts worries were not justified, Apple stock price started increasing gradually.

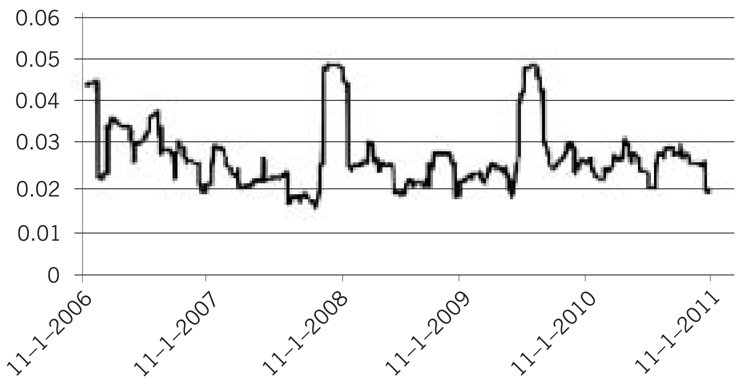

Figure 6.

Volatility of variance of Apple rates of return ("volatility of volatility") from 01.11.2002 to 01.11.2011

Source: Based on data from Yahoo!Finance service.

Figure 6. presents the so called "volatility of volatility". Periods of the highest volatility of variance fall on October and November 2008, as well as April and May 2010. However, in the whole period studied values are significantly higher than zero (taking into account the fact that the plot shows estimates of d a i l y variance volatility). Therefore, application of the GARCH model is intuitively justified.

3.2. Results

The binomial model often proves to be more useful than sophisticated pricing methods due to its simplicity. As mentioned, there are different methods to incorporate dividends in the model. However, there is well-founded suspicion that estimation errors are high enough to level the impact on valuation results of incorporating, usually relatively small to the value of the underlier, dividends in the model. The method of discounting the dividend to a given moment in time was used to verify this hypothesis empirically.

In case of unambiguously advantageous or disadvantageous economic situation it may be approximately assumed that dividend values paid at constant regular time intervals can be predicted with negligible error. Unpredictable changes on the market result in actual dividend amount departing slightly from its earlier values, as was the case of Bank of America from December 2008 on. For the simplicity of estimation the table of actual dividends was used as if all of them were known at the date of valuation.

Values of estimation errors, whether incorporating the dividend or not, are similar exact to 10^{-4} (see Table 2. in the Appendix). Due to a very small impact of incorporating the dividend in the binomial model, valuation without taking a dividend into account is preferred due to simplicity. Taking actual dividend values is the best possible approximate that a person pricing an option could have made. Therefore, the result is robust to uncertainty about dividend payments timing and departure of values from these assumed at the moment of pricing. If the dividend amounts were higher relatively to stock values the impact might have been higher. This, however, is a very rare case in practice. In further comparison the tree without dividend was used.

The program for GARCH-LSM pricing was written, using implementations of optimization algorithms and code fragments partly open for general use, i.a. fragments of program using Duan method for European options written by F. Rouah and implementation of the LSM method for the constant in time variance situation by T. Lipp, the author of articles on pricing European barrier bucket options [Hoppe, Lipp, 2011].

In the study, contrary to Stentoft [2005], where pricing using GARCH-LSM was conducted for weekly data, daily data was used. Additionally, the author used as a benchmark the constant volatility model, where options are priced using Monte Carlo method, due to difficulties with the binomial tree implementation for their data set. Here, it was shown that when discrete dividend is paid, its influence is often insignificant in comparison with pricing errors magnitude. Therefore, the basic binomial model was used for stocks paying relatively low dividend. For Apple options the underlier did not pay dividend in the period studied. Prices obtained from binomial model were, additionally to market data, used to assess GARCH-LSM pricing results.

To compute parameters in regression by the least squares method in one of the intermediate steps, the Householder decomposition was used. In very rare but existing cases, the number of paths for which option is in-the-money at the time for which the parameters are derived is lower than the number of basis polynomials (equal to 4). Then, for practical purposes (sufficient number of rows in the matrix), in the algorithm implementation to calculate the function of expected value of continuation all future price trajectories were used, not only in-the-money ones. When values are randomly generated by sampling from normal distribution, sometimes very large numbers may be obtained, which in case of GARCH-LSM may lead, in certain circumstances, to extremely high put option prices. Rare cases when values obtained from GARCH-LSM exceeded reasonable amount (the threshold of 1000 was chosen) were omitted in further analysis. It should be noted that without setting the top limit the method is not appropriate for put options pricing as it gives a very high average pricing error due to rare, extremely large results. The binomial model was estimated with the use of functions available in FinCAD.

In previous studies many different statistics describing estimation error were used to compare the quality of option pricing models [see for example: Stentoft, 2005]: Mean Error $\left(ME = \frac{\sum_{i=1}^{n} \hat{V}_i - V_i}{n} \right)$, Mean Absolute Error $\left(MAE = \frac{\sum_{i=1}^{n} \left| \hat{V}_i - V_i \right|}{n} \right)$, Root Mean Square Error $RMSE = \sqrt{\frac{\sum_{i=1}^{n} \left(\hat{V}_i - V_i \right)^2}{n}}$, Mean Percentage Error $(MPE) = \frac{1}{n} \sum_{i=1}^{n} \frac{\hat{V}_i - V_i}{V_i}$, Mean Absolute Percentage Error $\frac{\left| \hat{V}_i - V_i \right|}{¿V_i \vee ¿}$, Median Absolute Percentage Error $MAPE = \frac{1}{n} \sum_{i=1}^{n} ¿$ $\left(MdAPE = median\,value\,errors \frac{¿V_i^{\wedge} \vee ¿ \left| \hat{V}_i - V_i \right|}{¿} \right)$, where $\hat{V}_i$ stands for estimate of real fair market price and n is the number of priced options. Table 5. and 6. in the Appendix contain values of statistics mentioned above for call and put options for each of ten assets. All of the presented error measures unequivocally point to the fact that for the period studied the simple binomial model is superior. Using alternative methods to generate random values from normal distribution in Monte Carlo method improved slightly the quality of estimates obtained from the GARCH-LSM model. The reason for common case of MdAPE being equal to 1 is the fact that estimates obtained from the GARCH-LSM model were often equal to 0, when market price was a small positive number. High values of percentage errors, as compared to absolute errors, result from the existence of options of very low market price. Furthermore, for Apple options in the period of higher values of underlier's variance from October to November 2008 results of pricing with the binomial tree are also significantly closer to market price, both for call and put options (MdAPE around 99% for GARCH-LSM versus 19% for binomial tree).

Relatively high estimation errors for the GARCH-LSM model may be due to the fact that the number of paths used was not sufficient. Estimates obtained using the LSM method converge to the real option value as the number of trajectories generated with Monte Carlo technique goes to infinity [see: Zanger, 2009]. Consequently, prices are close to a correct value only for very a high number of paths, otherwise it may not yield accurate results. Moreover, in the LSM model a particular form of the function of expected value of continuation is assumed. In reality this function may take on other forms and so describing it as a linear combination of basis functions is only an approximation. It might be a possible source of errors in estimated option values. However, from the study conducted by the authors of the method—Longstaff and Schwartz [2001]—it follows that the choice of function space does not change price estimates significantly.

In estimating the GARCH model for each date four-year time series were used. For Apple options, changes in the price of the underlier at the moment of pricing are usually different than in the preceding period, as may be seen from the plot (Figure 5.). The result of that may be that the simulation made on the basis of the GARCH model estimates does not reflect price dynamics anticipated at the moment of pricing. In such a case, market data from a four-year period may paradoxically bring lower proportion of important information than data from the last 50 days before the moment of pricing used to calculate historical volatility utilized in binomial tree.

A hypothetical reason for the discrepancy between estimates and real market options price may be also under- or overestimation of the fair price by the investors on the market. In case of the studied options on stocks, especially Apple stocks, the probability of occurrence of such situation was limited by choosing the most liquid instruments. Supply, demand and price for such options adjust quickly and so in case the market price differed significantly from the fair price, there would be many investors trading and taking advantage of the situation to gain profit without risk. In turn the market price, in turn, would return to the fair price level after a short period.

Another reason for the low precision of GARCH-LSM estimates may be the fact that GARCH model dynamics in the specific form proposed by Duan is not necessarily a correct illustration of the underlier price movements. The solution in this case would be to use more sophisticated model specification, e.g. modifications allowing to incorporate asymmetrical reactions to innovations. This asymmetry, called the leverage effect, occurs when the influence on volatility of positive and negative shocks differs, i.e. the impact of negative shocks is higher and stock prices are negatively correlated with the volatility. Possibly, taking this effect into account would improve estimation results. Extended GARCH model specifications used in pricing are for example NGARCH and EGARCH used by Stentoft [2005] or GJR used in Piontek's

[2003] study of European options. Better proxies for Monte Carlo method may often be obtained also by variance reduction techniques. This, as well as possible optimization methods allowing for using higher number of price paths, constitutes an interesting problem for future research.

Using Monte Carlo simulation in combination with the least-square method, in which for each moment in time expected value function of the future payment is estimated depending on the current price of the underlier is relatively time consuming. Additionally, for each moment GARCH model parameters have to be estimated, which causes even higher computational complexity. When necessity occurs to price a huge number of options, as it is often the case in scientific studies or for investors making multiple transactions on a daily basis, the method becomes impractical, even taking into account huge computational capacities of computers and servers in modern financial institutions. GARCH-LSM in its basic implementation is ineffective even in comparison with other numerical techniques and this study illustrates the fact that when computational capacities of devices used for pricing are limited, estimates obtained with the usage of Monte Carlo method may be inaccurate, even compared to simple methods such as the binomial tree model. Even for the data range from time of financial crisis, when assets volatility was high, the binomial model proved better than GARCH-LSM. However, despite the imperfections mentioned above, simulation techniques have one major advantage, which is why it is worthy to study their properties and create various new modifications. Namely, in certain situations no other method may be used due to the specific features of the priced instrument e.g. for more complex derivatives with many underliers or a complex payment function dependent on the underlier's price history.

Table 7. and 8. in the Appendix presents MAE depending on option characteristics—"moneyness" ratio and time to maturity. Results indicate, that regardless of characteristics of the derivative estimates obtained from the binomial tree are, for Apple options in the studied period, more precise than proxies from GARCH-LSM model. For both methods, for call as well as for put options a clear growth trend is observed in estimates errors as the "moneyness" ratio decreases. The more an option is out-of-the-money, the less precise estimate of its value is. This phenomenon is often interpreted as a consequence of the fact that out-of-the-money options are treated as speculative assets and thus their market price often is not a reliable proxy of a fair price. In-the-money options, on the other hand, are often bought and sold by banks and other institutions possessing knowledge allowing proper pricing of the derivative [see for example: Sakowski. 2011]. High errors for options OTM and ATM with the shortest time to maturity may be interpreted and justified similarly. However, it should also be noted that for this type of options, price is usually relatively low and so even a small deviation from it results in a significant increase of relative error of estimate.

What is interesting about the valuation results is the occurrence of rather low errors for LEAPs (Long Term Equity Anticipation security)—long term options with time to maturity above one year. As far as these instruments are concerned, the stochastic character of factors often taken as constant, as risk-free interest rate, starts playing a more significant role. Here however, despite the fact that these variables were not considered, pricing errors are not high. An intuitive reason for that may be that changes in underlier price until maturity are practically not possible to predict but invariably not predictable in case of LEAP. This means that, at the moment of valuation, the events, which will occur over a long period of time remainings until maturity, are usually not observed. It is not possible to unambiguously infer about a possible LEAPs payment in a relatively distant future. Therefore, LEAPs price in subsequent periods will not change as significantly as for option with very short time to maturity, which, if not exercised, will soon expire, and so it is easier to predict. As a result the model price and market price are similar. Furthermore, LEAPs may be regarded as a long term investment and not an asset acquired for speculation. Consequently, similarly to ITM options, it is easier to estimate their value, as the market price is its good proxy. The overall trend was also for pricing errors to be higher for options with shorter time to maturity. This effect is mostly due to the high pricing errors for OTM and ATM options with short time to maturity.

4. Summary

The assessment results of two chosen option characteristics—"moneyness" ratio and time remaining to maturity—on pricing errors confirmed the intuitive hypothesis that the more in-the-money an option is the lower pricing errors occur. Interestingly, relatively low errors were obtained for LEAP options. A formal (e.g. using regression) investigation of dependence of options implied volatility on both mentioned characteristics could be an interesting extension of the study.

In order to verify the quality of a valuation method, a study based on sufficiently long time series of historical data has to be conducted. Computational capacities, software availability and especially access to data give financial institutions advantage over the private investor, not only when validating the model but also in everyday usage. However, outcomes presented in this paper show that commonly used simple pricing techniques, quicker and easier to implement, as binomial tree, yield similar or better results than more complex and advanced models e.g. GARCH-LSM. Adaptation and application of appropriate optimization techniques allowing for faster estimation of possibly more accurate proxies of option values using the GARCH-LSM method could be an important direction for further studies.

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Appendix

Table 1.

Number of observations

Symbol	Company name	Initial no of observations	After data cleaning	Call/Put
AAPL	Apple Inc.	645,298	266,091	123,443/ 142,648
AMD	Advanced Micro Devices Inc.	161,552	64,394	33,293/ 31,101
BAC	Bank of America Corp.	269,954	156,585	80,025/ 76,560
C	Citigroup Inc.	262,498	129,404	66,900/ 62,504
INTC	Intel Corp.	209,206	121,428	60,362/ 61,066
MO	Altria Group Inc.	169,890	88,618	42,150/ 46,468
MSFT	Microsoft Corp.	246,920	148,170	71,437/ 76,733
RIMM	Research In Motion Ltd.	258,598	245,805	118,825/ 126,980
WFC	Wells Fargo & Co.	233,338	127,204	58,664/ 68,540
YHOO	Yahoo Inc.	197,678	94,515	47,729/ 46,786

Table 2.

Estimation errors for models incorporating or not dividend payments for Bank of America options

Statistic	Model with dividend		Model without dividend	
	call	put	call	put
MAPE	3.302958	0.402738	3.302952	0.402756
MdAPE	0.545633	0.248310	0.545633	0.248439

Table 3.

The amount of call/put options by "moneyness" ratio (rows) and time remaining to maturity (columns).

	0–60		61–180		181–360		>360			
	call	put	call	put	call	put	call	put	call	put
Deep ITM	316	210	482	202	187	164	136	517	1121	1093
ITM	21026	8357	23731	9103	14253	6064	15538	7366	74548	30890
ATM	4586	4498	4804	4715	2287	2273	2619	2547	14296	14033
OTM	7617	22709	9050	30940	6220	20295	9070	21846	31957	95790
Deep OTM	83	55	199	194	155	267	1084	326	1521	842
	33628	35829	38266	45154	23102	29063	28447	32602	123443	142648

Table 4.

Dividends paid by Bank of America from 01.11.2006 to 01.11.2011

Ex-dividend date	Dividend value	Date of announcement of the dividend
2012–02–29	0.01	2012–11–01
2011–11–30	0.01	2011–11–18
2011–08–31	0.01	2011–08–22
2011–06–01	0.01	2011–11–05
2011–03–02	0.01	2011–01–26
2010–12–01	0.01	2010–10–25
2010–09–01	0.01	2007–10–28
2010–06–02	0.01	2010–04–28
2010–03–03	0.01	2010–01–27
2009–12–02	0.01	2010–09–28
2009–09–02	0.01	2009–07–21
2009–06–03	0.01	2009–04–29
2009–03–04	0.01	2009–01–16
2008–12–03	0.32	2008–10–06
2008–09–03	0.64	2008–07–23

2008–06–04	0.64	2008–04–23
2008–03–05	0.64	2008–01–23
2007–12–05	0.64	2007–10–24
2007–09–05	0.64	2007–07–25
2007–05–30	0.56	2007–04–25
2007–02–28	0.56	2007–01–24
2006–11–29	0.56	2010–06–25

Source: Bank of America, Dividend payment table, Investor relations.

Table 5.

Estimation errors for GARCH-LSM and binomial model for call options

Statistic		ME	MAE	RMSE	MPE	MAPE	MdAPE
AAPL	GARCH-LSM	–1.0410	4.3792	8.2516	–19.1245	22.7262	1.0000
	CRR	0.1910	0.2610	4.4595	–0.0675	0.5240	0.2274
AMD	GARCH-LSM	–0.8890	3.1557	4.9044	–21.1327	21.9855	1.0000
	CRR	0.1688	0.2431	0.3949	0.1690	0.2877	0.1775
BAC	GARCH-LSM	–1.6172	6.9233	10.1506	–59.3418	60.2235	1.0000
	CRR	–0.5829	0.8628	1.8041	–1.3851	3.3030	0.5456
C	GARCH-LSM	–2.1525	6.4247	9.6370	–88.9467	89.8067	1.0000
	CRR	–0.2822	0.5118	0.9463	–1.0229	1.3110	0.2172
INTC	GARCH-LSM	–0.1257	4.5613	6.1767	–21.2767	22.2323	1.0000
	CRR	0.0037	0.2643	0.4298	–0.0718	0.3073	0.1047
MO	GARCH-LSM	1.3675	6.4353	10.5882	–12.8609	14.0325	1.0000
	CRR	0.1654	0.4075	0.9555	–0.0375	0.2798	0.0973
MSFT	GARCH-LSM	0.6029	5.6004	7.5712	–25.6156	26.6885	1.0000
	CRR	0.0090	0.3520	0.5879	–0.1686	0.4186	0.1178
RIMM	GARCH-LSM	2.8168	27.4785	37.8479	–59.4226	60.3851	1.0000
	CRR	–0.1713	1.7054	2.9657	–0.2506	0.4137	0.1109
WFC	GARCH-LSM	1.0645	6.3243	8.3484	–14.1234	15.2306	1.0000
	CRR	–0.7841	1.0709	2.0246	–0.6603	0.7832	0.1501
YHOO	GARCH-LSM	–0.3446	5.0182	6.7148	–20.8976	21.8283	1.0000
	CRR	–0.0715	0.5077	1.2057	–0.1922	0.5256	0.1366

Table 6.

Estimation errors for GARCH-LSM and binomial model for put options

Statistic		ME	MAE	RMSE	MPE	MAPE	MdAPE
AAPL	GARCH-LSM	2.0051	6.8999	8.9264	–44.7263	46.1726	1.0000
	CRR	–0.1288	0.5320	1.1642	0.4429	0.4834	0.3631
AMD	GARCH-LSM	0.5726	2.5308	5.9228	–12.0725	13.1570	1.0000
	CRR	0.0514	0.1838	0.2921	0.2015	0.2757	0.1373
BAC	GARCH-LSM	0.2516	5.8266	7.7968	–33.7575	34.7614	1.0000
	CRR	–0.4572	0.7915	3.0635	–0.0400	0.4028	0.2484
C	GARCH-LSM	0.6039	5.7705	8.0897	–27.3490	28.3999	1.0000
	CRR	–0.1612	0.4620	0.8428	0.1454	0.3288	0.17488
INTC	GARCH-LSM	–0.0737	4.0549	5.5728	–30.5516	31.4930	1.0000
	CRR	0.1012	0.2909	0.4566	0.2727	0.3499	0.1964
MO	GARCH-LSM	–0.8449	5.0851	8.1561	–27.6226	28.4603	1.0000
	CRR	0.3884	0.5416	1.0535	0.4162	0.5351	0.4363
MSFT	GARCH-LSM	–0.3280	4.7702	6.5245	–40.8207	41.7507	1.0000
	CRR	0.0556	0.3524	0.6313	0.2610	0.3799	0.2182
RIMM	GARCH-LSM	–2.3325	25.0761	33.8264	–100.098	101.0052	1.0000
	CRR	–0.7537	1.5730	2.9397	0.0818	0.3095	0.1709
WFC	GARCH-LSM	–1.0602	6.6071	8.4187	–31.1641	32.0344	1.0000
	CRR	–0.4047	0.8213	1.5690	0.0823	0.4554	0.3095
YHOO	GARCH-LSM	0.5479	4.2141	6.9337	–22.2702	23.2886	1.0000
	CRR	–0.2186	0.4704	1.1237	0.0071	0.3994	0.1663

Table 7.

Mean Absolute Errors from GARCH-LSM and binomial model (CRR) for American call options on Apple stocks depending on the "moneyness" ratio (rows) and time remaining to maturity (columns)

		0–60	61–180	181–360	>360	
Deep ITM	GARCH-LSM	0.5019	0.3371	0.3079	0.5275	0.4018
	CRR	0.0105	0.0103	0.0107	0.0105	0.0104
ITM	GARCH-LSM	2.2381	2.2836	2.0364	2.2219	2.2106
	CRR	0.0787	0.0858	0.0711	0.1212	0.0884

ATM	GARCH-LSM	4.5927	4.8476	3.7263	3.2164	4.2876
	CRR	0.2422	0.5483	0.2110	0.2493	0.3414
OTM	GARCH-LSM	19.7634	11.3625	3.8706	3.0038	9.5343
	CRR	0.9721	0.6499	0.4533	0.3466	0.6023
Deep OTM	GARCH-LSM	17.7634	7.6976	7.1123	4.8297	6.1433
	CRR	3.9746	2.4387	0.9972	0.4773	0.9777
	GARCH-LSM	6.5508	4.7563	2.7176	2.6540	
	CRR	0.3123	0.2886	0.1936	0.2179	

Table 8.

Mean Absolute Errors from GARCH-LSM and binomial model (CRR) for American put options on Apple stocks depending on the "moneyness" ratio (rows) and time remaining to maturity (columns)

		0–60	61–180	181–360	>360	
Deep ITM	GARCH-LSM	0.0749	0.0856	0.0969	0.214	0.1460
	CRR	0.0187	0.0412	0.0433	0.0716	0.0516
ITM	GARCH-LSM	1.0749	0.8817	0.6532	0.5541	0.8110
	CRR	0.0418	0.0636	0.0763	0.1419	0.0789
ATM	GARCH-LSM	7.0400	3.0497	2.0152	1.4265	3.8665
	CRR	0.0274	0.1966	0.1930	0.2121	0.1446
OTM	GARCH-LSM	14.3229	11.1258	8.1002	2.6662	9.3134
	CRR	0.8992	0.7365	0.6807	0.6184	0.7363
Deep OTM	GARCH-LSM	30.7864	29.2243	11.2724	7.0012	15.0296
	CRR	1.0564	0.9970	0.9837	0.9905	0.9942
	GARCH-LSM	10.2603	8.2457	6.0545	2.0966	
	CRR	0.5849	0.5425	0.5156	0.4741	

Abstract **Simple is better. Empirical comparison of American option valuation methods**

Technique for American options valuation, combining Least Squares Monte Carlo with Duan's model under the assumption that the volatility of the underlier can be described by GARCH(1, 1) process, has been confronted with simple binomial tree model. Results of comparison of model outcomes with market prices for ten different CBOE-traded stock options indicate that simple binomial model is superior to sophisticated GARCH-LSM method. The results hold regardless of option characteristics—"moneyness" ratio and time to maturity. Incorporating dividend in binomial model does not significantly alter the valuation outcomes. Detailed analysis shows also that for each of the methods pricing errors grow as the "moneyness" ratio decreases.