Beliefs Revealed in Bayesian Equilibrium

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Abstract

Harsanyi in a series of papers (cf. [Harsanyi, 1967–1968]) introduced the Bayesian games framework for analyzing games with incomplete information. Part of its novelty consisted in replacing decision theoretically well motivated but unwieldy hierarchy of beliefs with a simple device, a type space, as a representation of incomplete information. The other half was the introduction of BE as the solution concept. The resulting framework—tandem of BE and type spaces—paved the way for the bulk of the contemporary literature on auctions, bargaining, insurance, principal-agent, moral hazard, rational expectations, repeated games, reputation, signaling etc.

However, type spaces and belief hierarchies are not interchangeable. Two states in two type spaces that formally correspond to the same hierarchy may support different action profiles in BE (see Bergmann and Morris [Bergmann, Morris, 2005], Battigalli and Siniscalchi [2003]). They model different environments, give different predictions of rational behavior under the very solution concept that completed Harsanyi's framework. Belief hierarchies are not expressive enough for the BE solution concept.[1]

This dissonance leads to immediate questions: What is the implicit feature of a type space that is crucial for BE? Since a type space is just a convenient thought model for representing beliefs of the agents (see e.g. Gul [1998]), does an explicit description that extends standard belief hierarchies exist? Moreover, belief hierarchies should not only provide a sufficient description of a state, but also have no redundant information. Is there any way of choosing "the right" belief hierarchies for the BE? Finally, given the new belief hierarchies, can we characterize the BE solution concept in terms of Bayesian rationality?

In this paper we answer these questions. We

- define new, more expressive belief hierarchies, denoted $\mathcal{L}^{\exists}$-belief hierarchies;

We show that the appropriate state description must include the beliefs about all possible action correlating signals, sunspots. Our $\mathcal{L}^{\exists}$-belief hierarchies are exactly geared to describe such beliefs in a consistent way. They ad-

[1] Or, BE is too strong a solution concept for the standard belief hierarchies. For the analysis along those lines, see Dekel, Fudenberg, Morris [2004].

ditionally include variables ranging in a type space over information partition cells, denoting signals. Their role is to describe the (co)relations between the beliefs of several agents. This kind of information is absent from standard hierarchies, which treat beliefs in a group as a set of unrelated beliefs of each agent, but is crucial for the equilibrium behavior. Furthermore, we

- prove that states in any type spaces agree on their $\mathcal{L}^{\exists}$-belief hierarchies if and only if they agree on the local BE (BE over some sub-type space) predictions, for every game.

To prove right to left implication we construct a countable collection of "canonical" finite games, each with a distinguished subset of action profiles. For every game and state m in some type space an action profile from the distinguished subset is played in BE at m if $\mathcal{L}^{\exists}$-belief hierarchy of m belongs to a certain set of hierarchies. Those sets of hierarchies are sufficiently fine that knowing BE predictions for every "canonical" game characterizes uniquely a $\mathcal{L}^{\exists}$-belief hierarchy.

We prove the other direction by showing that the local BE can be defined syntactically, explicitly within a $\mathcal{L}^{\exists}$-belief hierarchy. For any game and a state m in a type space an action profile can be played at m in local BE if a specific formula is part of the $\mathcal{L}^{\exists}$-belief hierarchy description of m. This formula expresses the fact that for some acting of agents on their beliefs (about signals and states of nature and beliefs about beliefs…) there is a common belief that everybody acts optimally. In other words we

- provide an epistemic justification for local BE;

The results can be interpreted as follows. Imagine that a researcher is facing an environment with uncertainty. For some fixed sets of agents and states of nature I and Θ, respectively, he can observe the action tuples chosen by the agents for any game $\Gamma : \Theta \times_{i \in I} A_i \to \mathbb{R}^I$. Suppose that agents act "rationally", where this is taken to mean that the action profiles are part of local BE for some type space. The description of the state of the world in terms of beliefs that he recovers is exactly a $\mathcal{L}^{\exists}$-belief hierarchy. Moreover, the "rationality" of local BE is warranted by its characterization as consistency with common belief in Bayesian rationality. The analysis is supposed to be in analogy with the original introduction of subjective probabilities by Ramsey [1931] and De Finetti [1931], in the (single agent) decision theoretic setting.

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